

# THE THEORY OF THE BOHR—WEISSKOPF EFFECT IN THE HYPERFINE STRUCTURE

F. F. Karpeshin

Mendeleev All-Russian Research Institute of Metrology,  
Saint-Petersburg, Russia

and

M.B.Trzhaskovskaya

Petersburg Nuclear Physics Institute, Kurchatov Research Center,  
Gatchina, Russia

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## Abstract

For twenty years research into the anomalies in the HF spectra was going in a wrong direction in fighting the Bohr—Weisskopf effect. As way out, we propose the model-independent way, which enables the nuclear radii and their moments to be obtained from the hyper-fine splitting. The way is based on analogy of HFS to internal conversion coefficients, and the Bohr—Weisskopf effect — to the anomalies in the internal conversion coefficients. It is shown that the parameters which can be extracted from the data are the even nuclear moments of the magnetization distribution. The radii  $R_2$  and (for the first time)  $R_4$  are obtained in this way by analysis of the experimental HFS for the H- and Li-like ions of  $^{209}\text{Bi}$ . The critical prediction is made concerning the HFS for the  $2p_{\frac{1}{2}}$  state. The moments may be determined in this way only if the higher QED effects are properly taken into account. Therefore, this set of the parameters form a basis of a strict QED test. Experimental recommendations are given, aimed at retrieving data on the HFS values for a set of a few-electron configurations of various atoms.

# 1 Introduction

Hyperfine structure e.g. of the  $D_1$  line of the  $^{133}\text{Cs}$  atom or that of the  $1s - 2s$  transition of the hydrogen atom play an important role in construction of the atomic clock (e.g. [1]). Experiments with the hyperfine structure allowed one to obtain more accurate value of the hyperfine structure constant  $\alpha$  — one of the fundamental constants. Worthy of notion is also the project of composing a reference point of frequency, founded on the few-electronvolt isomeric state of the  $^{229}\text{Th}$  nuclide [2], with unprecedented systematic shift suppression, allowing for the atomic clock performance with a total fractional inaccuracy approaching  $10^{-19} - 10^{-21}$ . Hyperfine structure is of especially great importance for the proper design of such clock [3]. At the same time, proper account of the hyperfine structure may radically change the estimated lifetime of the nuclear isomeric states [4].

From the viewpoint of theory, the hyperfine structure arises due to interaction of the electron magnetic moments with the nuclear magnetic moments. The interaction leads to a hyperfine splitting (HFS). That also depends on the electron density near the nucleus and other properties of the electron shell. In the first approximation, both nuclear and atomic factors act independently, which manifests itself in the *factorization* of the nuclear and electron parameters in the related formulae. However, precision measurements showed that the factorization is violated for the reason of the finite nuclear size. First of all, this is the Bohr-Weisskopf effect: finite distribution of the magnetic currents inside the nucleus. Study of the magnetic anomalies offers a way of pursuing the change of the effective nuclear radii within isotopic chains with addition of neutrons to the nucleus, which information is of primordial interest. That explains why a specific problem of a magnetic anomaly of the hyperfine splitting of the atomic structure became very topical for the past time [5, 6, 7, 8]. Nuclear-optical methods became very important in the research into the properties of rare or radioactive nuclides.

From the theoretical point of view, description of the hyperfine splitting is a challenging problem, as it needs a calculation of the atomic wavefunctions with the accuracy of  $\sim 10^{-4}$  [5, 9]. A convenient way of comparison of the data to theoretical calculations is offered by investigation of the HFS in few-electron atoms: H-, Li- or B-like ions. In these cases, the electronic wavefunctions can be calculated more reliably. Moreover, comparison to the theory can be used as a test for our capability of description of both the electronic structure of the atoms, as well as QED. In this aspect, the problem of the Bohr-Weisskopf effect in appearance of the magnetic anomaly is put forward in the first place. Existing method of its description through construction of the specific differences is however far from being appropriate and sufficient for this role. It is erroneous in principle, as it is clear from the internal conversion theory and is shown in the present paper.

An adequate way of solving this problem was proposed in refs. [10, 11, 12]. An analogy was noted between the internal conversion coefficients (ICC) and the HFS. It was shown that the HFS can be calculated as the limiting case of the ICC at the transition energy  $\omega \rightarrow 0$ . Test calculations were made for the H-like ions of  $^{209}\text{Bi}$ . Our present purpose is to show that the application of the classical theory of anomalous internal conversion [13, 14] gives us the proper tool for adequate description of the Bohr-Weisskopf effect, also offering the way of extracting the unique information about the nuclear radii. To this end, we consider the HFS for the  $s$ -electronic states. In this case, first, the effect is maximal, and second, the quadrupole effect of the nuclear deformation is absent.

## 2 Physical premises

In the case of the  $M1$  transitions, in the long-wavelength limit, the electromagnetic interaction of the nuclei are determined by the single formfactor, or the reduced current of the nuclear transition  $J(R)$ . We define it according to [15]:

$$\int \left( \mathbf{J}_0(\mathbf{R}) \mathbf{T}_{LM}^{(0)*}(\hat{R}) \right) d\Omega_R = \frac{C(I_f M_f LM | I_i M_i)}{\sqrt{2I_i + 1}} iJ(R). \quad (1)$$

Here  $\mathbf{T}_{LM}^{(\lambda)}$  are vector spherical harmonics:

$$\mathbf{T}_{LM}^{(\lambda)} = \sum_{\nu} C(1\nu L + \lambda M - \nu | LM) Y_{L+\lambda M-\nu}(\hat{R}) \xi_{\nu}, \quad (2)$$

with  $\xi_{\nu}$  being three basic unit vectors,  $L$  — the multipole order of the transition, and  $m$  — the corresponding magnetic quantum number. Below, we keep the name of the transition nuclear current for its radial component  $J(R)$ . This can be related to the radiative amplitude of the nuclear transition. Expressing the latter as follows

$$F_{\gamma} = -i \int J^{\mu}(\mathbf{R}) A_{\mu}^{*}(\mathbf{R}) d^3 R, \quad (3)$$

and substituting the expression for the vector potential in the case of transitions of the magnetic type [15, 16]

$$\mathbf{A}^{(LM)} = -2\sqrt{\omega} j_L(\omega R) T_{LM}^{(0)}(\hat{R}), \quad (4)$$

one arrives at the equation relating the transition nuclear current with the reduced transition magnetic momentum (or amplitude) of the nuclear transition [17]

$$\langle R J(R) \rangle \equiv \int_0^{\infty} J(R) R^3 dR = \sqrt{(L+1)/L} \langle I_2 | \mathcal{M}(ML) | I_1 \rangle. \quad (5)$$

Now, using in (5) the definition of the nuclear magnetic moment  $\mu$  [17]

$$\frac{e\hbar}{2M_p c} \mu = \sqrt{\frac{4\pi}{3}} \frac{C(II10|II)}{\sqrt{2I+1}} \langle I || \mathcal{M}(M1) || I \rangle, \quad (6)$$

one arrives at the expression

$$\langle RJ(R) \rangle = \sqrt{\frac{3}{2\pi}} (2I+1)(I+1)/I \frac{e\hbar}{2M_p c} \mu. \quad (7)$$

Relations (5) – (7) show that  $RJ(R)$  can be treated as the radial distribution of the magnetic currents over the nucleus. Therefore, baring in mind application for studying nuclear structure, one can advance the past expression to

$$\langle R^{j+} J(R) \rangle = \sqrt{\frac{3}{2\pi}} (2I+1)(I+1)/I \frac{e\hbar}{2M_p c} \langle \mu \rangle_j, \quad (8)$$

where  $M_p$  is the proton mass,  $\langle \mu \rangle_j \equiv (R_j)^j$  — the  $j$ -th moment of the radial distribution of magnetization over the nuclear volume.

Energy shift of the state with the total angular momentum  $F$  and its projection  $M$ , nuclear and atomic momenta  $I$  and  $j$ , respectively, can be viewed as an elastic amplitude of the  $M1$  internal conversion (IC) transition [12, 11, 10]. The amplitude is defined in terms of the interaction of the nuclear and electronic transition currents:

$$F_c = \int (\mathbf{J}(\mathbf{R}) \mathbf{s}(\mathbf{r})) \mathcal{D}(|\mathbf{r} - \mathbf{R}|) d^3r d^3R, \quad (9)$$

with the photon propagator

$$\mathcal{D}(|\mathbf{r} - \mathbf{R}|) = e^{i|\mathbf{r} - \mathbf{R}|}/|\mathbf{r} - \mathbf{R}| = 4\pi i \omega \sum_{l=0}^{\infty} j_l(\omega r_{<}) h_l(\omega r_{>}) Y_{lm}^*(\hat{R}) Y_{lm}(\hat{r}), \quad (10)$$

and the electron current

$$\mathbf{s}(\mathbf{r}) = e\bar{\psi}\boldsymbol{\gamma}\psi \equiv 4e\kappa\mathbf{j}(\mathbf{r}). \quad (11)$$

In (10),  $r_{<}$  ( $r_{>}$ ) designates the smaller (larger) of the  $r$  or  $R$ , and  $\hat{r}$  stands for  $\mathbf{r}/r$ . In turn,  $j_L(\omega R)$ ,  $h_L(\omega R)$  are the spherical Bessel and Hankel functions, respectively. In the case of a point-like nucleus,  $r_{<} = r$ ,  $r_{>} = R$ . These relations define the non-penetration (NP) model in the internal conversion theory. Within the NP model, interaction (9) with (10) is fully factorized with respect to the nuclear and electronic variables. The whole transition amplitude factorizes into the radiative nuclear amplitude and the remaining factor, independent of the nuclear variables. The latter just defines the ICC [15, 16]. With the account of the finite nuclear

size, such a factorization may only be achieved within the framework of the nuclear models, among which the most widely used became the models of surface (SC) and volume (VC) transition nuclear currents. On the physical ground, the SC model is more justified by the Pauli principle, which to a greater extent prohibits motion of the internal nucleons. For this reason, the SC model is the cornerstone of known tables [18]. It recommended itself as working well in the internal conversion theory. Difference of experimental ICC from their tabular values is called as anomalies. The anomalies are thus related to the effects of penetration of electrons into the nuclear area. We see that the values of thus determined anomalies depend on the nuclear model.

The diagonal matrix element of the IC interaction Hamiltonian  $H_c(L\mu)$  defines the HFS.  $H_c(L\mu)$  is spherical tensor of rank  $L$ ,  $L, \mu$  being the multipole order and its magnetic quantum number. In order to maintain succession with the internal conversion theory and previous papers [10, 12, 11], we will consider formulae in general case, and in the end we will pass to the mathematical limit.

Generally, amplitude (9) can be considered as a matrix element of  $H_c(L\mu)$ :

$$F_c = \langle b|H_c(L\mu)|a\rangle, \quad (12)$$

where  $|a\rangle, |b\rangle$  are the wavefunctions of the initial and final states of the atom. In the IC theory, they are usually characterized with the quantum numbers  $|IMjm\rangle$  of the nuclear and electronic spins and their projections on the quantization axis. On the other hand, the hyperfine shift  $S_{FIj}$  is determined by the diagonal matrix element in the  $|FMIj\rangle$  representation, with  $F$  and  $M$  being the total angular momentum of the atom and its projection. By making use of the Wigner-Eckart theorem, after straightforward algebra, the energy shift can be represented as follows [4]:

$$S_{FIj} = \sum_{\mu} \langle FMIj|H_c(L\mu)|FMIj\rangle = (-1)^{F-I-j} W\{IIjj; LF\} \langle Ij||H_c(L)||Ij\rangle. \quad (13)$$

The reduced two-bar matrix element in eq. (13) is defined as follows:

$$\langle I_2 M_2 j_2 m_2 | H_c(LM) | I_1 M_1 j_1 m_1 \rangle = \quad (14)$$

$$= \frac{C(I_2 M_2 LM | I_1 M_1) C(j_1 m_1 LM | j_2 m_2)}{\sqrt{(2I_1 + 1)(2j_2 + 1)}} \langle I_2 j_2 || H_c(L) || I_1 j_1 \rangle \quad (15)$$

Recoupling the angular moments in (9), (10), one can separate the angular variables [15, 16]

$$\begin{aligned} \frac{e^{i\omega|\mathbf{r}-\mathbf{R}|}}{|\mathbf{r}-\mathbf{R}|} (J(\mathbf{R})\mathbf{j}(\mathbf{r})) &= 4\pi i\omega \sum_{L\lambda M} \left( \mathbf{J}\mathbf{T}_{LM}^{(\lambda)*}(\hat{\mathbf{R}}) \right) \\ &\times \left( \mathbf{j}\mathbf{T}_{LM}^{(\lambda)}(\hat{\mathbf{r}}) \right) j_{L+\lambda}(\omega r_{<}) h_{L+\lambda}(\omega r_{>}). \end{aligned} \quad (16)$$

In the case of transitions of the magnetic type, the terms in (16) with  $\lambda = 0$  are relevant. Using definition (1) and integrating over the electronic variables, as shown in [15, 16], one arrives at the following expression:

$$\langle Ij || H_c(L) || Ij \rangle = \frac{16\alpha\pi\omega\kappa}{\sqrt{L(L+1)}} i^{l-l'+1} (-1)^{l'-j-L-\frac{1}{2}} W\{l'jlj; \frac{1}{2}L\} G, \quad (17)$$

$$G = \int_0^\infty J(R) j(r) j_L(\omega r_<) h_L(\omega r_>) R^2 dR r^2 dr. \quad (18)$$

Inserting (18) into (13), one arrives at the following expression for the hyperfine splitting  $w$  of the states with  $F = I + j$  and  $F = I - j$ :

$$w = -4e\omega\kappa \sqrt{\frac{6\pi j(2I+1)}{I(I+1)(j+1)}} G. \quad (19)$$

The fact that the hyperfine shift is generically related to the conversion amplitude is manifested in the limiting relation between the formulae for the ICC at  $\omega \rightarrow 0$  and for the HFS [12, 11]

$$\begin{aligned} S_{FIj} = & (-1)^{j+I-F} W(IIjj; LF) \frac{2}{(2L+1)!!} \left(1 + \frac{m}{M}\right)^{-3} \times \\ & \times \lim_{\omega \rightarrow 0} \sqrt{\omega^{2L+L} \alpha_\partial(M1)(2j+1)(2L+1)(L+1)/L} \times \\ & \times \langle I || \mathcal{M}(M1) || I \rangle, \quad (20) \end{aligned}$$

with  $L = 1$ , where  $m$  and  $M$  are the electron and nuclear masses, respectively. In (20),  $\alpha_\partial(\tau l)$  is the analogue of traditional ICC  $\alpha(\tau l)$  extended to the case where the conversion electron occupies a discrete electronic state [19, 20]. It has dimension of energy. Note that eq.(20) is exact in the sense that its dependence on the nuclear model can be fully related to the way of calculation of ICC. In the SC model, the magnetic current reads as follows [18, 21]:

$$J_{SC}(R) = D\delta(R - R_0), \quad (21)$$

$R_0$  being radius of the magnetic current. With  $R_0 = 0$ , this model includes the NP model. The constant  $D$  as determined from the normalization condition (7) reads as follows:

$$D = \sqrt{\frac{3}{2\pi}} (2I+1)(I+1)/I \frac{e\hbar}{2M_p c} \mu / R_0^3. \quad (22)$$

Then eq. (18) is factorized, resulting in

$$w = \frac{2(2I+1)}{I(j+1)} e\omega^2 \kappa \frac{e\hbar}{2M_p c} \mu i \mathcal{F}, \quad (23)$$

where  $\kappa$  is the relativistic quantum number, and the radial electronic matrix element

$$\mathcal{F} = \int_0^\infty j(r)X_L(\omega R)r^2 dr. \quad (24)$$

Eq. (23) can also be obtained from (20). In (24), the electronic transition current

$$j(r) = g(r)f(r). \quad (25)$$

In (25),  $g(r)$ ,  $f(r)$  are the large and small radial Dirac wavefunctions of the electron. Furthermore,  $X_L(\omega R)$  is the potential of the electronic transition.  $X_L = h_L(\omega R)$  within the NP model. In the SC model it reads

$$X_L(\omega R) = [h_L(\omega R_0)/j_L(\omega R_0)]j_L(\omega R). \quad (26)$$

The related ICC in eq. (20) reads as follows:

$$\alpha_d(\tau L) = |Q^{(L)}\mathcal{F}|^2, \quad (27)$$

$$Q^{(L)} = -4\kappa\sqrt{\frac{\alpha\pi\omega}{L(L+1)}}C(j - \frac{1}{2}L0|j - \frac{1}{2}). \quad (28)$$

### 3 The formalism

In any realistic nuclear model, after separation of the angular variables, the electronic and nuclear variables are mixed in the radial conversion matrix element (18). For the purpose of separation of the penetration effects, let us express the nuclear transition current in general form as

$$J(R) = J_{SC}(R) + [J(R) - J_{SC}(R)] \equiv J_{SC}(R) + J_p(R) \quad (29)$$

and, correspondingly, the matrix element (18) — in the form

$$G = G_{SC} + G_p. \quad (30)$$

Here  $G_{SC}$  is the factorized SC model contribution, and  $G_p$  bares information on the penetration effects. It reads as

$$G_p = \int_0^\infty J_p(R)R^2 dR [j_L(\omega R) \int_R^\infty j(r)h_L(\omega r)r^2 dr + \\ + h_L(\omega R) \int_0^R j(r)j_L(\omega r)r^2 dr] = \int_0^\infty J_p(R)Q(R) R^2 dR, \quad (31)$$

$$Q(R) = \int_0^R [h_L(\omega R)j_L(\omega r) - j_L(\omega R)h_L(\omega r)] j(r)r^2 dr. \quad (32)$$

In the nuclear area, the electronic wavefunctions are well represented by the Taylor series [22, 18]:

$$g(r) = a_0 + a_2 r^2 + \dots, \quad f(r) = b_1 r + b_3 r^3 + \dots \quad \text{for the } s \text{ states}, \quad (33)$$

$$g(r) = a_1 r + a_3 r^3 + \dots, \quad f(r) = b_0 + b_2 r^2 + \dots \quad \text{for the } p_{\frac{1}{2}} \text{ states}. \quad (34)$$

By making use of (33), (34) and asymptotic expansions for the spherical Bessel and Hankel functions, the electronic current in (31) can be put down as follows:

$$j(r) = c_1 r + c_3 r^3 + \dots \quad (35)$$

As a result, we arrive at the series expansion for  $Q(R)$  and, finally,

$$G = \frac{1}{\omega} \sqrt{\frac{3}{2\pi} (2I+1)(I+1)/I} \frac{e\hbar}{2M_p c} \mu \left[ \frac{\omega^2}{3} \mathcal{F} + \sum_{i=2,4,\dots} \frac{c_i}{i(i+3)} (R_0^i - (R_i)^i) \right]. \quad (36)$$

Therefore, by comparing experimental HFS with the theory for an atomic level, we obtain the root mean square radius  $R_2$  of the nuclear magnetization, and can unambiguously predict the HFS value for all the other  $s$ - and  $p_{\frac{1}{2}}$  levels. This parameter can associate with the effective nuclear radius which is usually extracted from the analysis of the data concerning the nuclear hyperfine anomalies. Furthermore, comparing experiment with theory for two levels, one can also find the fourth moment of the nuclear magnetization distribution and, correspondingly, predict more precisely the HFS value for the other levels, etc. Analysis of  $n$  levels provides us with the even nuclear magnetization moments up to the  $2n$ -th one, inclusively.

## 4 Results

Atomic calculations were performed by means of the package of computer codes RAINE [18]. Fermi nuclear charge distribution was supposed, with typical parameters. Higher order QED corrections were taken into account. A typical value of  $R_0 = 7.121$  fm was adopted for the radius of the nuclear magnetic current. The vacuum polarization potential and the electron selfenergy correction were allowed for as suggested in refs. [23, 24]. As a result, for the  $1s$  state, the value was obtained  $w_h = 5.107$  eV, in coincidence with the previous calculation [10]. Theoretical calculation should then be compared to the experimental value. For a long time, the value of  $5.0841_8$  eV was accepted [25]. A somewhat lower value of  $5.0863_{11}$  eV [26], though outside the range of the previous error bars, was obtained recently. Furthermore, the wavefunctions, calculated in this way, were used in order to fit the experimental data. To this end, we incorporated into the calculation the latest QED correction



[27, 28]  $\Delta w_{QED} = -0.0268$  eV. Being added to the basic “Coulomb” value of  $w^C = 5.0825$  eV of the bare nucleus, calculated by us taking into account pure electrostatic electron interaction with the nucleus, this results in the calculated HFS value of  $w_{th} = 5.0558$  eV, which is by 0.6% lower than the experimental value [26]. Then, dealing with the difference between the theory and the latter experimental value as shown previously, we achieve consensus with the experimental value for this level with the root-mean-square radius of the nuclear magnetization  $R_2^{1s} = 6.207$  fm. This is the only parameter which we can determine by comparison of theory with experiment for one level. We dwell on these details in order to show a possible range of scatter of the parameters and sensitivity of the model to them.

At the time being, there are available data concerning the HFS values for the upper states. The latest one obtained in the Experimental Storage Ring at GSI is  $0.79750(18)$  eV [26]. This is noticeably lower in comparison with the earlier measurement at the Lawrence Livermore National Laboratory [29] of  $0.820 \pm 0.026$  eV. Regarding theory, our calculation for a bare nucleus, with the Fermi charge distribution with usual parameters, and  $R_0 = 7.121$  fm yielded in the  $w_C^{2s} = 0.826$  eV. QED correction may be introduced according to [8] as  $\Delta E_{QED}^{2s} = -0.005$  eV. The most essential difference with the analysis of the  $1s$  case arises in the necessity of account of the electron-electron interaction  $\Delta E_{e-e}^{2s}$ . In the zeroth approximation, we allowed for that through the self-consistent Dirac-Fock method [18]. As a result, we arrived at the value of  $\Delta E_{e-e}^{2s} = -0.038$  eV. This a little exceeds the calculation [8], where all the terms up to the second order in the  $\alpha Z$  perturbation series were taken into account, and the electron screening correction of  $\Delta E_{e-e}^{2s} = -0.030$  eV was obtained. If we rely on the latter value, we arrive at the HFS of  $w_{th}^{2s} = 0.791$  eV. Fit to the experimental value of the HFS for this one level only, results then in the root-mean-square radius of the nuclear currents of  $R_2^{2s} = 6.275$  fm. We see that  $R_2^{1s} \neq R_2^{2s}$ , as we neglect the remaining terms in eq. (36) including  $R_4$  and higher moments. We then found that with account of the two radii,  $R_2$  and  $R_4$ , a better fitting the multipole moments of the magnetization is achieved if a value of  $w_{th}^{2s} = 0.792$  eV is used. Then solving the system of coupled algebraic equations (36) for the  $1s$ - (in the H-like ions) and  $2s$ - (in the Li-like ions) states results in the reasonably close value of  $R_2 = 6.118$  fm, and  $R_4 = 6.760$  fm. Note that the radii obtained satisfy a plausible relation  $R_2 < R_4$ .

As a test for the self-consistency of the method, we repeated the same calculations with another value of  $R_0 = 6.2$  fm, which is closer to the  $R_2$  obtained and therefore, should be more realistic, and provide with a better convergence of the series (36), as explained above. Indeed, we obtained as the starting values of  $w_{th}^{1s} = 5.079$  eV,  $w_{th}^{2s} = 0.796$  eV, which values are considerably closer to the experimental ones. As expected, by solving the system of two coupled equations (36) we arrived at the same value of  $R_2 = 6.12$  fm, and slightly corrected value of  $R_4 = 6.78$  fm,

Table 1: Representative results of calculation of the  $R_2$  and  $R_4$  magnetic radii. In the first column, the levels used for solving eqs. (36) are indicated. With both the levels involved in the analysis, the resulting  $R_2$  and  $R_4$  parameters hold in spite of essentially different starting values of the SC radius  $R_0$ .

level	$R_0$ , fm	$R_2$ , fm	$R_4$ , fm
1s	7.12	6.21	—
2s	7.12	6.27	—
both	7.12	6.12	6.76
both	6.20	6.12	6.78

differing by 0.3%. Now this was achieved already without additional varying the  $w_{th}^{2s}$  value. The scheme of calculation is illustrated in Table 1.

Let us see how this works for the  $2p_{\frac{1}{2}}$  state. Direct data concerning the HFS value for the upper states, are not available at the time being, to the best of our knowledge. Proceeding in the way similar to the 2s case above, we obtained the Coulomb base value of  $w^C = 0.28509$  eV for the  $2p_{\frac{1}{2}}$  state. Account of the  $R_2$ - and  $R_4$ -terms in eq. (36) lowers this value to 0.28498 eV. Allowing for the correction  $\Delta w_{e-e} = -27.19$  meV, owing to the electron-electron interaction, and the QED correction  $\Delta w_{QED} = -0.26$  meV [30], we arrive at the resulting value of  $w_{2p} = 0.25753$  eV. This may be compared to the theoretical value of 257.84(5) meV predicted in Ref. [30].

## 5 The Bohr-Weisskopf effect

In a series of papers, it was proposed to cancel the BW effect in linear combination of the HFS's of the two levels (specific difference) with a parameter  $\zeta$  ([7, 8] and refs. cited therein):

$$\Delta'w = w^{2s} - \zeta w^{1s}, \quad (37)$$

with the  $\zeta$  value of 0.16886. However, a model-dependent character of such relations was shown above. Eq. (37) is derived from the conventional Weisskopf model with constant nucleon wavefunctions inside the nuclei. Generally, the specific differences (37) are calculated with the same accuracy like each of the terms, that is  $\sim 1\%$ . It is worthy of noting here that the exact BW value is unobservable experimentally: this is a purely estimated value. The ultimate definition of the BW effect can be formulated as the difference between the experimental value of HFS and that calculated within the NP model:

$$b_\kappa = (w_{NP} - w_{exp})/w_{NP}. \quad (38)$$

The latter model would unambiguously and adequately predict the anticipated effect for the corresponding point-like nucleus. This model excludes influence of nuclear dynamics. At the same time, in this definition static nuclear properties like charge distribution influence the calculated NP value, together with QED corrections. We will not further separate out these effects in the definition (38). We summarize that the calculated NP values are 5.162, 0.811, and 0.25828 eV for the HFS values of the  $1s$ -,  $2s$ - and  $2p_{\frac{1}{2}}$  levels, respectively. These result in the BW effect of 1.73, 1.47 and 0.29 percent, respectively. It is these values of the BW effect which predetermines the difference between various models. We recall that the static nuclear properties contribute at the level of up to 10% to the ICC values in the case of heavy nuclei (e.g. [31] and refs. therein), that is even more than the BW effect (38).

Our above fit to the data on the HFS of the both  $1s$ - and  $2s$  levels results in the  $\zeta$  value of 0.1746, that is 3% higher than that cited above. This discrepancy is just what was fairly expected in the light of what was said above. We conclude that parameter  $\zeta$  in the specific difference (37) is neither observable, nor model-independent enough, to be predicted up to five digits by making use of only one or two experimental HFS values for the element.

## 6 Discussion

Consecutive theory of HFS, including the description of the anomalies caused by the Bohr—Weisskopf effect, is presented previously. That is founded on the classical theory of anomalous internal conversion. First, the theory shows the kind of information which can be obtained from studies of HFS'. This is even multipole moments of the nuclear magnetization distribution. They can be extracted by means of solving the system of coupled equations (36) for several electronic configurations, for which the experimental HFS' are available. Analysis of the data on the HFS in the H-like and Li-like ions of  $^{209}\text{Bi}$  resulted in the second- and, for the first time in internal conversion theory, the fourth-moment radii:  $R_2 = 6.12$  fm, and  $R_4 = 6.78$  fm. Dependence of the HFS on the  $R_4$  and higher moments of the magnetization distribution turns out to be a specific feature of HFS. Traditional IC theory assumes that there is only one parameter,  $R_2$ , which can be determined from comparison of theory with experiment. The same situation can be noted in the contemporary analysis of the anomalies in the HFS, observed within the isotopic chains: usually the only nuclear parameter, equivalent to  $R_2$ , is discussed (e.g., [6, 9]). Therefore, dependence of HFS on the higher magnetization distribution moments, demonstrated above, is an extension of the internal conversion theory. Furthermore, determination of higher radii from experiment is impossible in the absence of precision atomic calculations with proper account of higher order

QED corrections. Hence, the above way of analysis simultaneously offers a critical test of QED, otherwise impossible. The stringency of the test is only limited by the number of the electronic configurations of the atom of the same element for which data on the HFS are available. Our fit confirmed validity of the QED corrections calculated previously, within the framework of the latest experimental data.

As we saw above, the corrections at the level of  $\sim 1$  meV to the HFS of the  $2s$  level were crucial for treating the BW effect, which comprises  $\sim 0.1\%$  of the shift. This clarifies the sensitivity of the method to the parameters. The  $R_4$  value was obtained within the accuracy of about  $0.3\%$ , and the  $R_2$  value was most stable against fitting, not changed at all. This estimation shows more definitely the limits of the theory, as well as its abilities and prospects concerning future experimental data while they will be available.

These conclusions are derived by means of consecutive treatment of the BW effect, instead of fighting it, e.g. through combining the specific differences. The principal defect of the latter method is the loss of information concerning the nuclear structure, as this information is just conveyed *via* the BW effect. Another defect is that the recipe itself is erroneous, being founded on a model consideration. This means that for more than well two decades the research in this field has been actually going in the wrong direction, when fighting the BW effect instead of using it fruitfully for retrieving information from the data. Narrowing of the theme is also reducing the discussion about the nuclear anomalies to the single parameter — the effective nuclear radius, losing sight of  $R_4$  and other moments.

Prediction is also made for the HFS of the  $L2$ -electron, in which case data are not available up to date. Future experiments should manifest real relation between the discussed factors.

To summarize, the new nuclear-model independent way of treating the HFS is proposed, proceeding from the theory of anomalies which were first observed in ICC. The way is based on simultaneous analysis of experimental data concerning HFS for several electronic configurations of the atom. The method can be used as a stringent test of QED. Consequently, further development of the experimental basic research is needed, aimed at measuring the HFS on ions with various few-electronic configurations with a small number of electrons, where calculation of the electronic interaction is more feasible. For this purpose, the storage rings are suitable which are available or being in reconstruction e.g. at the Lawrence Livermore National Laboratory, GSI Darmstadt, Lanzhou in China. From the theoretical point of view, the QED corrections, nuclear recoil effect are of great importance for treating future data. A more consecutive way of incorporation of these effects into the calculation of the atomic wavefunctions looks quite feasible and should be pursued in further investigation.

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